The MOND paradigm: from Law to Theory

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Abstract

In this text I shall explain Tensor-Vector-Scalar gravity (TeVeS), and consider its worth in comparison to Modified Newtonian Dynamics (MOND) and RAQUAL. I shall attempt to give the reader an idea of the development of 'MOND' (Milgrom 1983) and highlight the features of its maturity into modern-day 'TeVeS' (Beckenstein 2004). I shall begin by presenting the need for an extension to gravitational theory from conventional Einstein Gravity. There are two general directions in which to proceed when regarding the spiral galaxy rotation curve problem. The first involves the need for the existence of Dark Matter, (on which I will be succinct), and the other requires a change (or extension) to gravitational theory. This text deals with the latter. After stating and then briefly explaining the *requirements* and then the *principles* (according to Beckenstein) to which such a theory must adhere, I will discuss six candidate theories in the chronological order in which they were first proposed, and indeed in the order of increasing complexity, and accomplishment. A disscusion of the ideas behind MOND will ensue, followed by RAQUAL, Phase Coupled Gravity, A theory with a disformally related physical metric, and finally TeVeS. I will continually remark on the successes of these theories, insofar as they fulfill the previously elaborated requirements and principles for a coherent theory of gravity.

Contents

1 Introduction: Where we stand, and our loss of predictive power in the astrophysical arena.

Einstein gravity made its debut effectively in 1915, and has been an incredible success for 20th century physics. The equivalence principle, general covariance, and achievement of reducing gravitational phenomenae to geometry has enjoyed a triumph. While the door was unlocked by Einstein in 1915, the relatavists of the 20th century truly have opened it and developed the initial theory a thousandfold, explaining countless astrophysical and cosmological occurences and verifying them with acceptable degrees of experimental precision.

In Einstein gravity the well-known Einstein-Hilbert action,

$$S_{EH} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}R \tag{1}$$

gives us the equations of motion of our world - expressing the way in which the curvature of our universe is affected by matter. The dynamics of how test particles (with or without mass) move through time, depends on this curvature. In practice this means the following: Given a mass distibution, we can use the equations of motion of our theory to solve for the metric tensor $g_{\mu\nu}$, and from that we can find the paths that test particles would follow.

A typical such example of this would be the lensing of light, or how the stars in a galaxy move around the galactic center. In many cases however, a *weak field limit* approximation suffices for such calculations - simplifying the calculations considerably. In the weak field case, the effective potential that a test particle feels due to a point-like mass is given by

$$\phi = -\frac{GM}{r}$$

This is indeed equivalent to the Newtonian potential. This approximation is valid for describing the orbits of many astrophysical bodies. The planets around the sun, the moons around the planets, the satellites around the earth. For a body in equilibrium, by simply equating the centripital force with the gravitational *weak field* force, given by

$$F = \frac{GMm}{r^2},$$

we can calculate the tangential velocity of the orbiting body. We obtain

$$\frac{GMm}{r^2} = \frac{mv^2}{r} \tag{2}$$

giving for the tangential velocity $v \propto \frac{1}{\sqrt{r}}$. Note that to do this we have of course assumed the Newton's Second Law to be true.

Upon consideration of the data, we find that this law applies well to the aforementioned systems. Uranus, having an orbital radius around the sun 19 times that of the Earth, does indeed have an orbital velocity of $\frac{1}{\sqrt{19}}$ of that of the Earth. But what about larger scales? We would expect the same to apply for the stars orbiting the galactic center. It seems however, that our predictive power suffers a blow at this scale. Indeed the galactic rotation curve does *not* follow this law. The stars orbiting the galactic centers are observed to have a *constant* velocity with increasing distance from the center. The gravitational force does *not* fall off with increasing distance as we are used to on our scales. Something must change. The important question is where do we look? How do we make a change to avert this. Do we simply not understand galaxies properly - ie, is our understanding of the *system* lacking; or is it our model of gravity (or perhaps our concept of inertia) that needs a makeover?

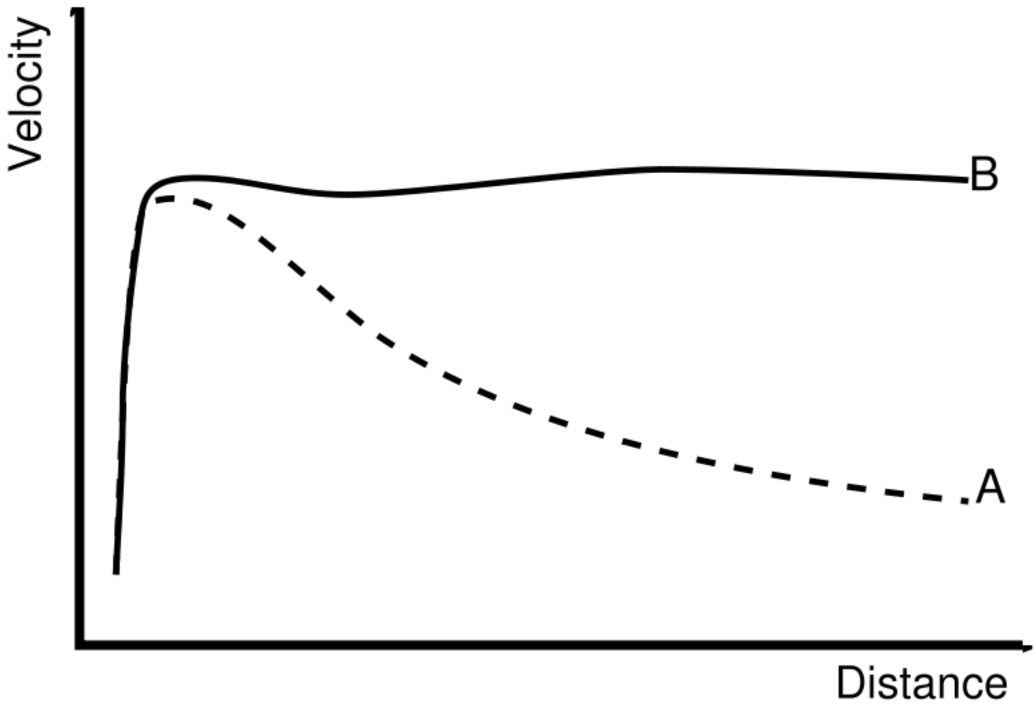

Figure 1: This diagram shows the tangential velocity of the stars in a spiral galaxy against their distance from the galactic center. The dotted curve (A) is what we expect from the Newtonian (or from the general relativity weak field limit) case. The velocity falls off hyperbolically with the distance. (Note that the rotation curve is not hyperbolic close to the center, since we are actually dealing not with a point-like mass system, but with a mass distribution.) The full curve (B) shows the *observed* rotation curve. From the galactic bulge onwards, the tangential velocity of the stars does not decrease with increasing distance from the center.

2 Dark Matter

2.1 Proposal

The dark matter suggestion argues that a substantial quantity of the matter in the universe is nonbaryonic, that is, invisible. This matter is suspected to be in the form of a massive particle, which is weakly interacting. This means that any interaction that this particle would have with any type of baryonic matter exhibits an extremely low cross-section (likelihood of occuring). Thus traces of particles of this type are extremely hard to detect. Photons for instance, the quanta of energy which is emitted or absorbed in atomic and nuclear interactions, are detectable by our eyes and our telescopes. Photons are the footprints of these interactions. Due to the weakly-interacting nature of dark matter particles, we have little hope of any direct detection of this kind.

The swiss astronomer Fritz Zwicky [13] noted as early as 1933 that the observed mass in the Coma Cluster was simply too low to be able to account for the fact that the galaxies within the cluster move in the same potential well - keeping the system bound. This was the first suggestion that there existed invisible matter that was unaccounted for by direct telescopic observation. Using the fact that the observed dynamics of the cluster indicated a bound system, he was able to estimate the average mass within the galaxies to be approximately 160 times greater than what was expected for their luminosities.

The designation of dark matter particles as *weakly interacting* is somewhat ambiguous. The adverb *weakly* refers primarily to the low cross section. However, it is also suspected that dark matter particles interact only via the *weak* fundamental force. Of course, this is in addition to its interaction via the gravitational force. (Were it not, dark matter would not concern us in the least for our purposes.) And because we currently also have no hope of detecting the quantum of the gravitational force, the graviton, we are forced to look for indirect evidence of dark matter. Namely, a change in gravitational dynamics of a system which we suspect to contain dark matter. Indeed, a proposed solution to the galactic rotation curve problem is the dark matter hypothesis. It is surmised that the matter content in galaxies is actually far greater than what we observe. From the Tully-Fisher relation, astonomers are able to calculate the mass of a star from its luminosity. On the galactic scale however, 'summing' the masses of the stars in a single galaxy does *not* reach the suspected mass levels of the entire galaxy. It is believed that the galaxy sits in the middle of a gigantic *halo* of dark matter.

2.2 What reasons do we have to believe that the mass of a galaxy is far greater than what our telescopes tell us?

There are three primary pieces of evidence which support this. The first comes from the theory of galaxy formation. In order for galaxies to form, massive gas clouds need to cool. Models of galaxy formation *without* the presense of dark matter simply have densities too low to be able to cool quickly enough. The universe is approximately 10^{10} years old. Galaxy formation theory shows that without dark matter, pre-galactic gaseous structures would need at least 10^{11} years to form into galaxies. Thus, we would not expect galaxies to exist for at least another one hundred billion years.

Our second piece of evidence comes from weak lensing. Deflection of light by stars is indeed well predicted by Einstein gravity. However, at the galactic scale, gravitational theory using the luminous matter as the total mass of the galaxy is unable to account for the observed dynamics. The lensing of light from stars behind galaxies shows that the lensing of light by such galactic lenses carries a deflection angle greater than what would be expected from the total mass of the luminous matter. The question once again arises - is our understanding of the galactic *system* lacking (do we need to add dark matter), or is our theory at fault on such scales?

Finally, the third argument for dark matter comes from the galactic rotation curve problem itself. Were we to add dark matter halos to galactic systems, we find an effective gravitational potential of the form $V \propto \ln r$. If we differentiate this with respect to r to find the force, and then equate it with the right hand side of (2) (the centripital force), we indeed find a constant rotation curve.

Although these arguments contend the case for dark matter within or around galaxies themselves, there is strong cosmological evidence for the existence of dark matter from cosmological considerations. The dark matter hypothesis is largely accepted by the scientific community to be correct. Dark matter is indeed strongly built into the theory of structure formation, and is even linked to inflationary theory. It remains to be understood however, exactly what type of particle this invisible matter composes. This task we leave to the particle physicists. From an astrophysical perspective, it is the dynamics of galaxies which interest us - and how to explain them. Let's leave the dark matter proposition, and attempt to construct a theory of gravity which would account for the above phenomenae without invoking dark matter.

3 What kind of features should we expect in a resonable theory of gravity?

A modification of physical laws that would explain the observed galactic rotation curve is no easy task. In the formulation of such, we stand to possibly lose key requirements (which need to be satisfied in order to be taken seriously), and principles (which are intrinsic to any gravitational formulation). For instance: a modification that 'fixes' our problem might well introduce others. We would require the overall features of such a theory to still contain the bulk of traditional gravity, so as to keep the previous achievments of gravitational theory 'in-line' with what we expect. (We need such phenomenae to not only exist in our theory, but remain unchanged enough - ie reducible to Einstein Gravity only in a certain limit.) MOdified Newtonian Dynamics (MOND), which we will consider soon, satisfies some of these requirements while explaining the galactic rotation curve anomaly.

Beckenstein [4] lists some important requirements to which any theory of gravity should asscribe in order to be taken seriously. This is essentially a list of phenomenae which have so far been successfully explained and modelled using conventional theories of gravity. We do not want to lose these successes.

3.1 Requirements

These requirements essentially force the theory to be Einsteinian at certain scales depending on the corresponding phenomenon.

- *Agreement with conventional theories at extragalactic scales* The weak field approximation should be recovered in the region of low-mass distrubution. We need our theory of gravity to approach Einstein gravity in a certain low-mass limit, which automatically means we will hit the Newtonian gravity regime on some further limit.

- *Agreement with the conventional phenomenology of gravitational lensing* We require the bending of light by large masses to be present in this theory. We would expect the deflection angles to be slight departures from those given by Einstein Gravity - perhaps even no such departure in the weak field limit.

- *Agreement with the dynamics of the Solar System* An array of measurabe phenomenae in the Solar System should be predictable by the theory, and should concur with experimental tests. These include the deflection of light rays, the time delay of radar signals, and the precessions of the perihelia of the inner planets; all of which are accounted for by Einstein gravity.

- *Agreement with binary pulsar tests* Binary pulsar systems are used to measure such effects as relatavistic time-delay, periastron procession and orbit decay due to energy loss via gravitational radiation. It is in this arena that we test the *strong field limit* of the theory, and thus the arena in which we are most likely to see departures from conventional Einstein gravity. Indeed, our theory will face its toughest challenge here, as the orbital decay of the binary pulsar system PSR1913+16 due to energy loss via gravitational waves has been remarkably well predicted by Einstein gravity.

- *Concurrence with cosmological essentials* The Friedman equations for this theory should predict the same cosmological phenomenology that has been developed and confirmed from Einstein gravity. These include the Hubble expansion, the timescale for various eras, the existence of a microwave background, and light element abundances from primordial nucleosynthesis which has been strongly verified by modern cosmological theory.

While this list contains essentially the *observable* features of the universe which we shun from losing in our theoretical framework, a theory of gravity needs to comply to the following fundamental physical *principles*, that we require to hold true. A theory exhibiting all of the abovementioned *requirements*, which does not contain the following principles, can be said to be only an *effective theory* - one which has been constructed from the top down, and not the other way round. MOND for example, is an example of an effective theory, as it is built upon *none* of these principles. RAQUAL and TeVeS however, are indeed built from the bottom-up, and so manage to at least partially or completely satisfy these principles.

3.2 Principles

• *The Action Principle* The equations of motion of our theory must *be able* to be directly derivable from an Action. This way, we guarentee the conservation laws of energy, and angular and linear momentum, which we of course require.

• *General covariance* The equations of our theory must be written in the language of traditional Einstein gravity. The action must be relativistic so that Poincaré invariance is not lost, and that when we '*look closely enough*' at any region of our spacetime manifold, we recover Special Relativity.

• *Equivalence principle* The theory must be *metric* - the *matter* or *nongravitational* (like electromagnetism, the weak force etc.) laws of physics should be expressible simply by rewriting their actions by replacing the Lorentzian metric with gravitational one.

• *Causality* To preserve the logical consistency of the theory, we require that the theory remain causal. This means that the maximum possible speed of propagation of any measurable field or of energy and linear and angular momentum is not superluminal. Luminal speed is the speed which is invariant under Lorentz transformations. Because Maxwell's equations are invariant under Lorentz transformations, this *luminal speed* is the speed of light.

4 Modified Newtonian Dynamics

4.1 Basic Equations

MOND was originally proposed by Mordehai Milgrom in 1981 [1]. The idea presented is essentially one which directly modifies Newton's second law $\vec{F} = m\vec{a}$. Indeed, this relation has only been verified in the high-acceleration regime. Milgrom proposed the following modification:

$$\vec{F} = m\mu\left(\frac{a}{a_0}\right)\vec{a} \tag{3}$$

where μ is an unspecified function however, with the following behaviour:

$$\mu\left(x\right) = x \text{ if } \mid x \mid \ll 1$$

$$\mu\left(x\right) = 1 \text{ if } \mid x \mid \gg 1$$

and a_0 is a constant. At normal accelerations, ie when the ratio $\frac{a}{a_0}$ is large, we recover Newton's Second Law. However, when the accelerations are small, and the ratio is less than one, we enter the MONDian regime, in which newton's second law is essentially *stronger* than expected. This means that an applied force yields a greater relative acceleration in the MONDian regime for a body with a mass m than in the newtonian regime.

The introduction of a change to inertial law of this type serves as to modify the dynamics of a body orbiting another body so as to flatten the rotation curve. The newtonian gravitational force is unchanged:

$$F = \frac{GMm}{r^2}$$

To find the acceleration, we use our modified second law:

$$F = \frac{GMm}{r^2} = m\mu\left(\frac{a}{a_0}\right)a$$

Assuming the equality of inertial and gravitational mass (although admittedly the concept of both might appear to be rather fuzzy under a modification of Newton II), we obtain

$$\frac{GM}{r^2} = \mu\left(\frac{a}{a_0}\right)a$$

In orbital equlibrium, the gravitational acceleration is equal to the centripital acceleration $a = \frac{v^2}{r}$. We get

$$a = \frac{GM}{r^2}\left[\mu\left(\frac{v^2}{ra_0}\right)\right]^{-1} = \frac{v^2}{r}$$

It is clear that we recover the standard non-flat, hyperbolic rotation curve at high accelerations.

Let's assume that we are in the MONDian regime, dealing with low accelerations. This corresponds to

$$\mu(x) = \frac{a}{a_0} = \frac{v^2}{ra_0}$$

giving for the tangential velocity

$$v = \sqrt[4]{GMa_0}$$

In the galactic context, provided we are far enough from the center - ie sufficiently far away such that the accelerations are low enough, we find a constant rotation curve. In its departure from the newtonian regime, inertia is weakened, and gravity is seen to be effectively *strengthened.* This relation is dependent upon the fundamental constant a_0, which sets the threshold at which MOND takes over. Of course, we expect its value to be extremely low. Were it not, this previously described consequence would be apparent on scales smaller then the galactic. In his original paper, Milgrom sets the value of this constant at $a_0 = 1.2\ 10^{-10} ms^{-2}$.

And so we see that by modifying Newton's second law, we obtain a result which fixes our original problem. Equation (3) is however not the starting point to MOND. Indeed, it is the gravitational potential which deserves to be modified. It is therefore Poisson's equation which needs to be considered. The poisson equation gives us the newtonian gravitational potential for an arbitrary mass distribution.

$$\nabla^2\Phi_N = 4\pi G\rho$$

The appropriate MONDian modification to this equation is

$$\nabla\cdot\left[\mu\left(\frac{\mid\nabla\Phi_M\mid}{a_0}\right)\nabla\Phi_M - \nabla\Phi_N\right] = 0$$

where Φ_M is the MONDian potential, and Φ_N is the newtonian. The unspecified function μ is dependent not on a ratio of accelerations $\frac{a}{a_0}$, but on a ratio of relative *forces.* The function μ however retains the same behaviour: $\mu(x) = 1$ for $x \gg 1$ (Newtonian limit), and $\mu(x) = x$ for $x \ll 1$ (MONDian regime). Let's consider the MONDian regime:

$$\nabla\cdot\left[\frac{\mid\nabla\Phi_M\mid}{a_0}\nabla\Phi_M - \nabla\Phi_N\right] = 0 \tag{4}$$

Now since the divergence of a curl is always zero, $(\nabla\cdot\nabla\times\vec{\xi} = 0)$

$$\frac{\mid\nabla\Phi_M\mid}{a_0}\nabla\Phi_M - \nabla\Phi_N = \nabla\times\vec{h}$$

The vector field $\vec{h}$ is unspcecified and unknown, yet null whenever the mass density distribution is either spherical, cylindrical or planar. In this case, we obtain for the effective MONDian acceleration:

$$g_M = g_N\sqrt{\frac{a_0}{\mid g_N\mid}}$$

where g_N is the acceleration corresponding to the newtonian field.

This reformulation of gravitational acceleration has been largely successful in describing the motion of stars in galaxies and the dynamics of clusters of gravity. As far as predictive power of this type goes, MOND has been largely successful.

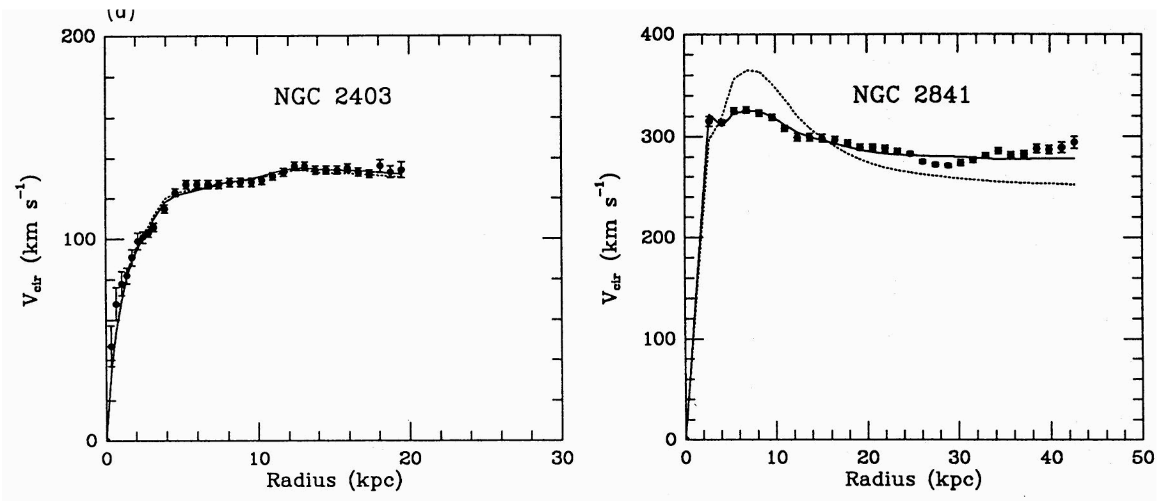

Figure 2: A good agreement between an observed rotation curves and the MONDian law (with Milgrom's constant $a_0 = 1.2\ 10^{-10} ms^{-2}$) for two galaxies.[9]

4.2 Shortcomings and Criticism

Perhaps the foremost reason undermining the credibility of MOND, is the way in which MOND is built. It is simply an effective theory, modifiying a simple gravitational law in an *an hoc* manner so as to produce a desired result: the flattening of the galactic rotation curve. The noun 'theory' is certainly no suitable label for such handiwork. No underlying 'reason' is proposed which would lead to this change in Newton's Second Law. The apparent formulation seems languid in justification, and ambitious in consequence. MOND should not be considered as a theory, but rather as a successful phenomenological scheme for which an underlying explanation (or theory) is necessary.

The formulation of gravitational theory changed dramatically one hundred years ago: The advent of the qualities of General covariance, an action formulation, and the inclusion of a *dependent* temporal component into the theory. These are central to the construction of a theory of gravitation. MOND overlooks these themes entirely, jumping straight to the solving of a problem at a 'high level', with no regard to these meritable principles. In doing so, it finds itself in troubled waters.

• *Intertial vs. Gravitational Law* The MONDian modification to Newton's Second Law fails to address the question of whether or not the modification is actually a *gravitational* one, or simply an *inertial*. Indeed, the question is irrelevent when considering the acceleration of a body under in a gravitational field alone. In this case, the final result does not change. Consider however, the acceleration of a massive charged particle, moving in both a gravitational and electric field. Do we 'extend' our MONDian modification to apply to the resultant acceleration due to an applied electric force - effectivly labeling MOND not as a gravitational modification, but rather as an inertial one?

Or on the other hand, do we withold this 'extension' to the electric force - keeping the modification in the gravitational term in our equation of motion - resulting in normal motion for charged particles, no matter their acceleration. The problem is essentially one of 'adding forces' together in an equation of motion. In a multi-body system, in which there is more than one term on the right hand side, MOND gives no obvious perscription of how to proceed. If the MONDian modification is indeed inertial - we have no problem. We calculate the acceleration as we normally would. But if forces beyond just the gravitational act on the body, we are unable to use the standard MONDian law (3).

• *Loss of energy conservation* The formulation of any theory without an action spells ruin for conservation laws. Indeed, MONDian dynamics violates conservation of energy, conservation of momentum, and conservation of angular momentum.

• *Loss of GR* The successes of the description of a spacetime *fabric* are all lost. MOND incorperates no metric tensor - the concept of curvature holds no place. This means that all the traditional successes of General Relativity are lost. The above mentioned requirements are not fulfilled. MOND is incomplete. Phenomenae such as perihilion precession in non-weak fields, gravitational redshift and gravitational lensing are unaccounted for.

• *Satellites orbiting satellites paradox* MOND fails to explain why a MONDian body (a star for instance orbiting a galactic center) feels the MONDian acceleration, while the star's components (gas, orbiting satellits) belong to the newtonian regime. Since these compenents are themselves part of the galactic system, why do they not share this MONDian acceleration?

These shortcomings should not manifest in disdain towards this theory; rather, MOND should not be regarded as a theory. The MONDian *effect*, exhibits results which explain the galactic rotation curve anomaly. We should indeed use what we have learnt, and attempt to develop a fully fledged theory of gravity: one which contains not only the Newtonian potential in a weak field limit (as all theories of gravity do), but one which contains in addition a Newtonian *low acceleration* limit - such that the MOND regime emerges. From MOND, we now take the next step towards AQUAL. We shall build on the previously elabourated upon ideas, and eventually present TeVeS - currently the most successful 'MONDian' candidate. We can expect the fomulation of such *complete* theories of gravity to be less naive in approach, and more complex in formalism.

5 AQUAL

We shall now begin the construction of our first attempt towards a full reconstruction of MOND. The starting point for any theory is an action. In this section, I shall rederive the results of the previous section from an action. Before writing down the action for MOND, we shall consider the action for Poisson's equation, and derive the corresponding equations of motion.

5.1 The Poisson Lagrangian

The Newtonian action is:

$$\mathcal{S}_N = -\int d^3x \left[\rho\phi_N + \frac{1}{8\pi G}\left(\nabla\phi_N\right)^2\right] \tag{5}$$

we shall work in Einstein notation, with

$$\left(\nabla\phi_N\right)^2 \equiv \partial_\mu\phi_N\partial^\mu\phi_N$$

To find the equations of motion, we vary with respect to our field, ϕ_N. The corresponding equations of motion are given by the Euler Lagrange equation:

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_N)}\right) - \frac{\partial\mathcal{L}}{\partial\phi_N} = 0 \tag{6}$$

where

$$\mathcal{S} = \int d^3x\mathcal{L}$$

Now

$$\frac{\partial \mathcal{L}}{\partial \phi_N} = -\rho$$

and

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_N)} \right) = \frac{\partial}{\partial x^\mu} \left(\frac{1}{4\pi G} \partial_\mu \phi_N \right)$$

yielding

$$\nabla^2 \phi_N = -4\pi G \rho$$

and so from writing down the action, we have thus been able to derive the equation of motion for Newtonian gravity. Let's take another step forward, and do the same for MOND.

5.2 AQUAL Action formulation

The action we propose is:

$$\mathcal{S}_{AQUAL} = -\int d^3x \left[\rho \phi_N + \frac{a_0^2}{8\pi G} \; f \left(\frac{(\nabla \phi)^2}{a_0^2} \right) \right] \tag{7}$$

with the function f exhibiting the following behaviour:

$$f(x) = x = \frac{(\nabla \phi)^2}{a_0^2} \text{ for } x \gg 1$$

(high acceleration ie newtonian) and

$$f(x) = \frac{2}{3} x^{\frac{3}{2}} = \frac{2}{3} \left(\frac{\nabla \phi}{a_0} \right)^3 \text{ for } x \ll 1$$

(low acceleration ie MONDian) This theory has been named 'AQUAL' from 'AQUAdratic Lagrangian'. Let's derive the corresponding equations of motion, once again using (6).

$$\frac{\partial \mathcal{L}_{AQUAL}}{\partial_\mu \phi} = -\rho$$

$$\frac{\partial}{\partial x^\mu} \left(\frac{\partial \mathcal{L}_{AQUAL}}{\partial (\partial_\mu \phi)} \right) = \frac{\partial}{\partial x^\mu} \left(\frac{1}{4\pi G} \partial_\mu \phi \frac{\partial f'}{\partial (\partial_\mu \phi)} \right)$$

Acting on the bracket with our differential operator gives two terms. We demand however that the gradient of the field vanishes on the boundary.

$$|\nabla \phi| \to 0 \text{ as } x \to \infty$$

Thus we arrive at our equation of motion

$$-\rho = \frac{1}{4\pi G} \partial^\mu \partial_\mu \phi \; \mu \left(\frac{\nabla \phi}{a_0} \right) \tag{8}$$

where we have made the identification $\mu(x) = f'(x^2)$, with the unspecified function μ being identical to the one from the previous section. Since

$$\rho = \frac{1}{4\pi G} \nabla^2 \phi_N$$

we can rewrite (8):

$$\nabla \cdot \left[\partial^\mu \phi \; \mu \left(\frac{\nabla \phi}{a_0} \right) - \partial^\mu \phi_N \right] = 0 \tag{9}$$

Thus we have reached (4). (9) reduces to the simple MOND formula (3) under cases of high symmetry. If this symmetry is violated, our dynamics are slightly modified. Using (7) as our Action ansatz, we have derived the MONDian form to the Poisson equation. We have not done much - simply reexpressed the previously ideas elaborated upon using an action formulation. In our reevaluation of the same law, we have managed to procure a worthy *principle* to our theory: by deriving the equations of motion from an action, we have assured ourselves that the theory contains the necessary conservation laws of linear and angular momentum, and energy. These follow from the symmetry of the lagrangian under spacetime rotations and translations. Linear and angular momentum and energy arise from these symmetries in the form of Noether currents (which are always conserved), with which we make the associated identifications.[1]

Nonetheless, much is still to be done. AQUAL is not formulated with a metric tensor, and general covariance is still absent. We have not yet attempted to include traditional Einstein gravity into our formulation. For this, we turn to Relativistic AQUAL.

6 RAQUAL

In the inclusion of Einstein gravity, we are obliged to introduce the metric tensor $g_{\alpha\beta}$, which quantifies the curvature of space-time, and which uniquely describes the motion of free particles. In Einstein gravity, the metric tensor is the solution to the Einstein field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$$

(which are directly derivable from the Einstein Hilbert Action (1)) with a given energy-momentum distribution encapsulated in $T_{\mu\nu}$. Our aim is to include the MONDian change, while completely keeping this Einstein-Hilbert distinctiveness present.

6.1 Extending Einstein gravity

We shall proceed as follows: We introduce a second metric, an *effective* or *physical* metric, which describes the manifold upon which free test particles move. This metric is conformally related to the above mentioned metric $g_{\alpha\beta}$, which is a solution to the Einsten field equations.

$$\tilde{g}_{\alpha_\beta} = e^{2\psi} g_{\alpha\beta} \tag{10}$$

By writing our physical metric in this way, and by demanding that the lagrangians of all matter fields be written using this same metric, we ensure the universality of free fall, and thus the principle of equivalence. ψ is a real scalar field which we shall use to include the MONDian phenomenology. This theory therefore contains two fields. The first is a tensor field, ie. the metric tensor $g_{\alpha\beta}$, in which traditional Einstein gravity is contained. The second field is a scalar field, which will incorporate our MONDian ideas. This will be done soon by choosing an appropriate action for ψ.

Let us concentrate on the dynamics of our spacetime. It is in the dynamics that we see the motion of particles under the influence of gravity. The significance of the metric tensor in describing the curvature (and therefore the motion through time and space) of the particles is given by the following action.

$$\mathcal{S}_m = -m \int \sqrt{-\tilde{g}_{\alpha\beta} dx^\alpha dx^\beta} \tag{11}$$

where m is the mass of the particle. We shall consider the null (massless) case later.

Note that the conformal metric has been used, which depends both on the metric tensor, and the scalar field. Writing down this action in terms of these variables gives simply

$$\mathcal{S} = -m \int e^\psi \sqrt{-g_{\alpha\beta} dx^\alpha dx^\beta}$$

[1] See [10] for the full proof.

Because the geodesic equation is derived from the variation of (11) with respect to the metric tensor [8], and then demanding $\delta\mathcal{S}_m = 0$, we can state that particle motion in our theory is *geodesic* with respect to $\tilde{g}_{\alpha\beta}$. This means that a particle sitting on the manifold described by $\tilde{g}_{\alpha\beta}$ will move on a trajectory described by the geodesic equation; only, with the connection coefficients calculated from the conformal metric. We shall proceed exactly in this manner in order to prove the presence of the primary MONDian feature (3) in this theory.

6.2 Geodesic behaviour in the Quasistatic limit

The geodesic equation reads

$$\frac{d^2x}{d\tau^2} + \tilde{\Gamma}^{\mu}_{\alpha\beta}\frac{dx^{\alpha}}{d\tau}\frac{dx^{\beta}}{d\tau} = 0$$

In order to explore the behaviour in the MONDian regime, we consider first the Newtonian limit to the metric tensor. We decompose the metric tensor into a sum of the Minkowskian part and a small perturbation.

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \tag{12}$$

In addition, because the velocities we are dealing with are small (nonrelativistic), we consider the *quasistatic* case: the movment of a body through time is far 'faster' than the movement through space.

$$\frac{dt}{d\tau} \gg \frac{dx^i}{d\tau}$$

where i runs over the spatial coordinates. This simplifies our geodesic equation to

$$\frac{d^2x^{\mu}}{dt^2} + \tilde{\Gamma}^{\mu}_{00}\left(\frac{dt}{d\tau}\right)^2 = 0$$

The expression for the Christoffel symbols in terms of the metric tensor is given by

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2}g^{\mu\lambda}\left(g_{\lambda\alpha},_{\beta} + g_{\lambda\beta},_{\alpha} - g_{\alpha\beta},_{\lambda}\right)$$

We need the Christoffel symbols in the conformal frame.

$$\tilde{\Gamma}^{\mu}_{\alpha\beta} = \frac{1}{2}e^{-2\psi}g^{\mu\lambda}\left[\left(e^{2\psi}g_{\alpha\lambda}\right),_{\beta} + \left(e^{2\psi}g_{\beta\lambda}\right),_{\alpha} - \left(e^{2\psi}g_{\alpha\beta}\right),_{\lambda}\right]$$

The components relevant to the quasistatic case are

$$\tilde{\Gamma}^{\mu}_{00} = \frac{1}{2}e^{-2\psi}g^{\mu\lambda}\left[\left(e^{2\psi}g_{0\lambda}\right),_{0} + \left(e^{2\psi}g_{0\lambda}\right),_{0} - \left(e^{2\psi}g_{00}\right),_{\lambda}\right]$$

By demanding that both our fields remain static, the temportal derivatives vanish:

$$\begin{aligned}\tilde{\Gamma}^{\mu}_{00} &= -\frac{1}{2}e^{-2\psi}g^{\mu\lambda}\left(g_{00}e^{2\psi}\right),_{\lambda} \\ &= -\frac{1}{2}g^{\mu\lambda}g_{00},_{\lambda} - g^{\mu\lambda}g_{00}\psi,_{\lambda}\end{aligned}$$

Writing out our metric in terms of the Minkowski plus a small perturbation (12) and dropping the higher order terms,

$$\tilde{\Gamma}^{\mu}_{00} = -\frac{1}{2}\eta^{\mu\lambda}h_{00},_{\lambda} - \eta^{\mu\lambda}\eta_{00}\psi,_{\lambda} - \eta^{\mu\lambda}h_{00}\psi,_{\lambda} - h^{\mu\lambda}\eta_{00}\psi,_{\lambda}$$

Let's consider the Christoffel symbols corresponding to the spatial components of the geodesic equation. Using the Minkowski metric to raise indices, we get

$$\tilde{\Gamma}^{i}_{00} = -\frac{1}{2}h_{00}^{,i} - \eta_{00}\psi^{,i} - h_{00}\psi^{,i} - h^{i\lambda}\eta_{00}\psi,_{\lambda}$$

In our choice of signature, $\eta^{00} = -1$ and $\eta^{ii} = 1$, giving

$$\tilde{\Gamma}^i_{00} = -\frac{1}{2} h^{,i}_{00} + \psi^{,i} - h_{00}\psi^{,i} + h^{i\lambda}\psi_{,\lambda} \tag{13}$$

It is at this point that we make the corresponding association of $h_{\alpha\beta}$ to the newtonian potential. So that we hit the correct newtonian behaviour when $\psi = 0$, we make the following identification: $h_{\alpha\beta} = -2\phi_N$ where ϕ_N is the newtonian potential as calculated from the Poisson equation. This means the last two terms in (13) cancel. Finally writing down the spatial components of the geodesic equation, we are left with

$$\frac{d^2\vec{x}}{dt^2} = -\nabla\left(\phi_N + \psi\right) \tag{14}$$

Thus our acceleration is modified. This departure from the newtonian acceleration is contained in our scalar field ψ.

6.3 Implementing the MONDian phenomenology

With the form of (14) in mind, we now consider the field equation for the scalar field itself. We start with an action ansatz of

$$\mathcal{L}_\psi = -\frac{1}{8\pi G L^2}\tilde{f}\left(L^2 g^{\alpha\beta}\psi_{,\alpha}\psi_{,\beta}\right) \tag{15}$$

Once again we have a function $\tilde{f}$, whose behaviour is unknown *apriori*. L is a constant with dimension of length. In the case $\tilde{f}(y) = y$, (15) is simply the lagrangian density for a real scalar field. $\mathcal{L}_\psi$ is however in general aquadratic.

Let's consider the case of a point source with physical mass M at $r = 0$. The corresponding lagrangian density is

$$\mathcal{L}_\psi = -\frac{1}{8\pi G L^2}\tilde{f}\left(L^2\left(\nabla\psi\right)^2\right) - \psi M\delta(r)$$

Compare this to the AQUAL lagrangian (7). It is clear that for the same mass M, ψ corresponds to ϕ as computed from (9), provided we take $\tilde{f} = f$ and $L = \frac{1}{a_0}$. When $|\nabla\psi| \ll |\nabla\phi_N|$ then the equation of motion (14) reduces to $a = -\nabla\phi$, and our MONDian phenomenology is recovered. We have to be careful in the newtonian limit though. In the $|\nabla\psi| \gg a_0$ regime, $\tilde{f}(y) = y$ so $\psi = \phi_N$. So (14) seems to exhibit an acceleration twice the normal value. This simply means however, that the measurable newtonian gravitational constant G_N is twice that of the constant G which appears in the Lagrangian density (15).

6.4 Achievments and Inadequacies

So RAQUAL contains, in addition to full Einstein gravity, the appropriate Newtonian and MONDian limits. In moving from AQUAL to RAQUAL we have accomplished much. Our *principles* for a coherent theory of gravity are almost completely fulfulled. Our theory is covariant, it has been successfully formulated with an action, and the equivalence principle holds. One important (unfortunately intrinsically required) principle is absent: causality. It has been shown [10], [4] that waves of the scalar field ψ can propagate superluminally. This is essentially a manifestation of the in-general aquadratic form of the lagrangian.

The array of phenomenae, successfully predicted and measured by Einstein gravity are present in RAQUAL. The explanation for this is essentially two-fold. Firstly because the recipe for RAQUAL included the Einstein-Hilbert action, and secondly because the inclusion of the scalar field was done in such a way so as not to disturb the relativistic (high acceleration, strong field) limit. In hitting traditional einstein gravity, we are unfortunatly not wholly triumphant. If the MONDian proposition is correct, and dark matter is not needed to explain the rotation curve flatness, then it means that mass in galaxies is completely due to baryonic (visible) matter. By the same token, weak field lensing deflection should be completly accounted for by this matter. Observations show however, that the light from distant stars is deflected more that what would be expected by treating the lens as a purely baryonic object. If the MONDian paradigm is to be taken seriously, deflection increase of light by the Baryonic matter component of galaxies needs to be explained (or rather included) into our theories.

7 Progress in theories leading up to TeVeS

In order to solve these last two gripes with RAQUAL, we need to massage our theories further.

7.1 Phase Coupled Gravity

Phase coupled gravity (hereafter PCG) was envisaged in order to resolve RAQUAL's acausal scalar field propagation problem. Like RAQUAL, it retains the conformal link (10) between the Einstein metric and the physical one. Our geodesics therefore remain unchanged with respect to (14), and thus we recover the MONDian as well as the relativistic limit. (Provided of course, suitable dynamics for ψ) The change that PCG makes is in the scalar field lagrangian. PCG introduces a second scalar field A coupled to ψ in the following manner:

$$\mathcal{L}_{\psi,A} = -\frac{1}{2}\left[g^{\alpha\beta}\left(A_{,\alpha}A_{,\beta} + \eta^{-2}A^2\psi_{,\alpha}\psi_{,\beta}\right) + \mathcal{V}\left(A^2\right)\right] \tag{16}$$

where η is a parameter, and $\mathcal{V}$ is a real valued function of the square of the scalar field A which we have yet to choose. Varying (16) with respect to A yields the equation of motion:

$$A_{;\alpha}^{;\alpha} - \eta^{-2}A\psi_{,\alpha}\psi^{,\alpha} - A\mathcal{V}'\left(A^2\right) = 0 \tag{17}$$

and varying with respect to ψ gives

$$\left(A^2 g^{\alpha\beta}\psi_{,\beta}\right)_{;\alpha} = \eta^2 e^{\psi} M\delta\left(r\right) \tag{18}$$

where we have included a point mass M at the origin.

Field propagation is kept luminal because the first derivatives are only in quadratic form. Note that the derivative of the function $\mathcal{V}$ is with respect to A^2. Let's now specify $\mathcal{V}$ and attempt to recover the sought after MONDian phenomenology. The MONDian limit corresponds to an almost Minkowski metric, reducing the covariant derivatives to partial derivatives; and a weak ψ. Thus (17) and (18) reduce to

$$\nabla^2 A - \eta^{-2}A\left(\nabla\psi\right)^2 - A\mathcal{V}'\left(A^2\right) = 0 \tag{19}$$

and

$$\nabla\cdot\left(A^2\nabla\psi\right) = \eta^2 M\delta\left(r\right) \tag{20}$$

The AQUAL phenomenology is recovered for small values of the parameter η. Using the two equations of motion for our fields, we are able to exactly relate the two scalar fields to one another. Let us choose an appropriate $\mathcal{V}$, and see how the dynamics unfold. We choose ([4], [7])

$$\nabla^2 A + \mathcal{V}\left(A^2\right) = -\frac{1}{3}\epsilon^{-2}A^6$$

This is a non-linear partial differential equation, coupled in ψ and A. The spherically symmetric solution to this system is

$$A = \left(\frac{\kappa\epsilon}{r}\right)^{\frac{1}{2}} \tag{21}$$

and

$$\frac{d\psi}{dr} = \frac{\eta\bar{\omega}}{4\kappa r} \tag{22}$$

where

$$\bar{\omega} \equiv \frac{\eta M}{\pi \epsilon} \tag{23}$$

$$\kappa \equiv 2^{\frac{3}{2}} \left(1 + \sqrt{a + 4\bar{\omega}^2}\right)^{\frac{1}{2}} \tag{24}$$

Using geodesic equation derived from the conformal (or effective metric) (14), we are able to see the final result of our MONDian regime dynamics. Substituting our solution for ψ into (14), we obtain for the radial acceleration

$$a_r = -\frac{GM}{r^2} - \frac{\eta^2 M}{4\pi\epsilon\kappa r} \tag{25}$$

The first term (25) is ϕ, our newtonian term, and the second is the effective MONDian term. Thus the newtonian $\frac{1}{r^2}$ term vies against a $\frac{1}{r}$ term. For large enough M ($M \gg 10^7 M_{\odot}$), $\kappa \approx \frac{1}{2}\sqrt{\bar{\omega}}$, and the $\frac{1}{r}$ force scales with $M^{\frac{1}{2}}$, and begins to dominate when the Newtonian acceleration drops below the scale

$$a_0 \equiv \frac{\eta^3}{4\pi G\epsilon}$$

And consequently, we have identified Milgrom's constant a_0 for this theory - the scale at which MONDian features begin to take over.

In conclusion, we see that PCG retains the primary MONDian features, and due to the $\frac{d\psi}{dr} \propto r^{-1}$ form, flattens the galactic rotation curve. Note however that we recover MOND only in the high-mass regime. This is however perfectly suitable for rotation-curve modelling on the scale of galaxies or galactic clusters.

TeVeS, the theory which we are working towards, inherits the coupled non-dynamical scalar field A, and assumes small η, which corresponds to the limit in which A becomes nondynamical. In our formulation of PCG, we have managed to ensure the subluminality of propagation of all the fields. As with RAQUAL however, PCG does not explain the increased deflection of light by galaxies. Null geodesics remain unaffected by the scalar field as the null solutions correspond all to $ds^2 = 0$, which is invariant in both metric frames due to the form of the conformal transformation.

7.2 Theories with disformally related metrics

In order to solve the under-deflection of light, we have no choice but to modify the relation between the Einstein and Effective metrics. Because null geodesics correspond to

$$ds^2 = \tilde{g}_{\alpha\beta} dx^\alpha dx^\beta = e^{2\psi} g_{\alpha\beta} dx^\alpha dx^\beta = 0$$

the conformal transformation does nothing to affect null paths. It was therefore suggested [12] to exchange the *conformal* relation with a *disformal* one of the following form:

$$\tilde{g}_{\alpha\beta} = e^{-2\psi} \left(\mathcal{A} g_{\alpha\beta} + \mathcal{B} L^2 \psi_{,\alpha}\psi_{,\beta}\right) \tag{26}$$

where $\mathcal{A}$ and $\mathcal{B}$ are functions of the invariant $g^{\mu\nu}\psi_{,\mu}\psi_{,\nu}$, and L is a constant of units of length. Assuming that light is metric on the manifold as described by our physical metric, we see that the scalar field ψ in (26) does indeed play a role in affecting the path of light. Hypothetically therefore, we would be able to find an appropriate choice for the ψ field lagrangian, which would appropriate an increase light deflection, as well as reproduce the required RAQUAL/MOND features. It was shown however, [6], that in order to ensure causality, by demanding the luminal propagation of both light *and* gravitational waves, the sign of the function $\mathcal{B}$ needs to be changed oppositely to that required to increase the magnitude of light deflection. Thus theories with disformal relations of the form (26) are doomed either with too-little-light-deflection or acausality.

A different disformal relation has been suggested [11] which breaks free of this conundrum. Sanders suggested a disformal transformation of the form

$$\tilde{g}_{\alpha\beta} = e^{-2\psi} g_{\alpha\beta} - 2\mathcal{U}_\alpha \mathcal{U}_\beta \sinh(2\psi) \tag{27}$$

where $\mathcal{U}_\alpha$ is a constant 4-vector which points in the time-direction. This modification essentially *stratifies* the theory - stretching spacetime in preferred directions. The theory is said to hold up strongly under solar system tests. The problem introduced here however, is that due to the addition of a *preferred* direction of $\mathcal{U}_\alpha$, our theory is not frame independant. Theories of the disformal type bequeath TeVeS the stratification contrivance.

8 TeVeS

Tensor-Vector-Scalar gravity, as the name suggests, incorporates all three types of fields into the theory. It builds on the ingrediants we have used in constructing the RAQUAL, PCG, and stratified theories. As with all these theories, an effective metric $\tilde{g}_{\alpha\beta}$ is present in the theory, which we use to construct the matter field lagrangians of our theory, and upon whose manifold test particles move. Like RAQUAL and the successor theories which followed, the deviation from classical general relativity is contained in this effective metric, which is related in some way to the gravitational scalar or vector fields included in our theory. With the correct choice of these scalar and vector fields, we are able to produce the wanted phenomenological features in certain limits, such as the flattening of the galactic rotation curve, and increased deflection of light by gravitational lenses.

As we have seen, by attempting to procure these features into a theory of gravitation, coherency is sacrificed. An edifice of snags must be overcome: acausality and loss of general covariance for instance. In our attempt to retain consistency of our theory under such stringent *requirements*, we are forced to massage our theory so as to produce the desired consequences without losing logical reliability. This massaging generally involves the addition of more fields, witty choices for lagrangians, and an appropriate relationship between the physical and the einstein metric such that we achieve the appropriate dynamics for our theory without forgoing consistency. Indeed, TeVeS continues in this regard with the expected increase in complexity.

8.1 The Action terms

TeVeS' action consists of four parts:

$$\mathcal{S}_{TeVeS} = \mathcal{S}_g + \mathcal{S}_v + \mathcal{S}_s + \mathcal{S}_m \tag{28}$$

The first three terms contain the lagrangians for our gravitational fields, and are thus written in terms of the Einstein metric. The final term in (28) contains the matter fields of our theory, written down using the conformal metric. We have yet to specify the relationship betwen the two metrics.

The form of the geometrical part of the action is as we expect:

$$\mathcal{S}_g = (16\pi G)^{-1} \int d^4x \sqrt{-g} R \tag{29}$$

where g is the determinent of the Einstein metric $g_{\alpha\beta}$. As with the previous theories, we include this term in order to keep TeVeS sufficiently close to Einstein gravity.

The term for the scalar field part of the action reads

$$\mathcal{S}_s = -\frac{1}{2} \int d^4x \sqrt{-g} \left[\sigma^2 h^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} + \frac{1}{2} G l^{-2} \sigma^4 F\left(kG\sigma^2\right) \right] \tag{30}$$

where $h^{\alpha\beta} = g^{\alpha\beta} - \mathcal{U}^\alpha \mathcal{U}^\beta$ ($\mathcal{U}^\alpha$ is the TeVeS dynamical vector field), F is a free dimensionless function, analogue to PCG's potential function $\mathcal{V}$, k is a dimensionless constant, and l is a constant of units of length. σ is a nondynamical (there is no kinetic term) scalar field analagous to PCG's A, coupled to a dynamical scalar field ϕ. Because there are no kinetic terms for σ, upon variation (6) with respect to σ, we obtain an equation of motion that directly relates σ to the invariant $h^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta}$. When we substitute this into (30), we obtain an action of the AQUAL form - which produces our MONDian dynamics. The new term $-\sigma^2 \mathcal{U}^\alpha \mathcal{U}^\beta \phi_{,\alpha}\phi_{,\beta}$ serves so as to eliminate superluminal propagation of the ϕ field.

TeVeS features in addition to a dynamical tensor and scalar field, a dynamical vector field (which we used in the definition of $h^{\alpha\beta}$) with dynamics given by the following action:

$$\mathcal{S}_v = -\frac{K}{32\pi G}\int d^4x\sqrt{-g}\left[g^{\alpha\beta}g^{\mu\nu}\mathcal{U}_{[\alpha,\mu]}\mathcal{U}_{[\beta,\nu]} - 2\frac{\lambda}{K}\left(g^{\mu\nu}\mathcal{U}_\mu\mathcal{U}_\nu + 1\right)\right] \tag{31}$$

where K is a dimensionless constant, and λ is not a constant; but a lagrange multiplier which will enforce a normalization condition on the vector field, namely

$$g^{\alpha\beta}\mathcal{U}_\alpha\mathcal{U}_\beta = -1 \tag{32}$$

keeping the vector $\mathcal{U}_\beta$ timelike.

Because all test particles must move on our physical metric, the final term in our lagrangian, the term which facilitates the inclusion of the matter content to the theory, must be written in terms of the physical metric $\tilde{g}_{\alpha\beta}$. The matter action we obtain in the usual way - by writing the appropriate matter theory's lagrangian in flat coordinates, only replacing the Minkowski metric with the effective metric, replacing partial derivatives with covariant derivatives taken w.r.t. $\tilde{g}_{\alpha\beta}$ and multiplying by the Lorentz invariant factor $\sqrt{-\tilde{g}}$. This scheme essentially 'smears' the theory over the curved manifold described by the metric $\tilde{g}_{\alpha\beta}$. For an arbitrary field f^α, the general form of the curvilinear action is

$$\mathcal{S}_m = \int d^4x\sqrt{-\tilde{g}}\mathcal{L}\left(\tilde{g}_{\mu\nu}, f^\alpha, f^\alpha_{;\mu}\ldots\right) \tag{33}$$

Finally, we must state the relationship between the physical and the Einstein metric. The physical metric in TeVeS is *stratified.* We obtain it by *stretching* the Einstein metric in the spacetime directions orthogonal to $\mathcal{U}^\alpha \equiv g^{\alpha\beta}\mathcal{U}_\beta$ by a factor $e^{-2\phi}$, while *shrinking* the Einstein metric by the same factor in the direction parallel to $\mathcal{U}^\alpha$:

$$\tilde{g}_{\alpha\beta} = e^{-2\phi}\left(g_{\alpha\beta}\mathcal{U}_\alpha\mathcal{U}_\beta\right) - e^{2\phi}\mathcal{U}_\alpha\mathcal{U}_\beta \tag{34}$$

$$= e^{-2\phi}g_{\alpha\beta} - 2\mathcal{U}_\alpha\mathcal{U}_\beta\sinh\left(2\phi\right) \tag{35}$$

8.2 Equations of Motion

Let's consider the [4] equations of motion resulting from the variation of the action $\mathcal{S}_{TeVeS}$ with respect to the appropriate fields.

8.2.1 Metric equations

The variation of $\mathcal{S}_g$ with respect to $g^{\alpha\beta}$ gives [8] the left hand side of the standard Einstein field equation

$$\delta\mathcal{S}_m = \left(16\pi G\right)^{-1}G_{\alpha\beta}\sqrt{-g}\delta g^{\alpha\beta} \tag{36}$$

where $G_{\alpha\beta}$ denotes the Einstein tensor of $g_{\alpha\beta}$.

The variation of the matter part of the action with respect to $\tilde{g}^{\alpha\beta}$ yields

$$\delta S_m = -\frac{1}{2}\tilde{T}_{\alpha\beta}\sqrt{-\tilde{g}}\delta\tilde{g}^{\alpha\beta} + \ldots \tag{37}$$

where the ellipses denote the variation of the matter fields, and $\tilde{T}_{\alpha\beta}$ denotes the *physical* energy-momentum tensor given by

$$\tilde{T}_{\alpha\beta} = \tilde{\rho}\tilde{u}_\alpha\tilde{u}_\beta + \tilde{p}\left(\tilde{g}_{\alpha\beta} + \tilde{u}_\alpha\tilde{u}_\beta\right) \tag{38}$$

where $\tilde{\rho}$, $\tilde{p}$, and $\tilde{u}_\alpha$ are the density, pressure and four-velocity respectively, all expressed in the physical metric.

After variation of the metric with respect to the scalar and vector terms in (28), we obtain the TeVeS analogue to the Einstein field equation:

$$G_{\alpha\beta} = 8\pi G \left[\tilde{T}_{\alpha\beta} + \left(1 - e^{-4\phi}\right) \mathcal{U}^\mu \tilde{T}_{\mu(\alpha}\mathcal{U}_{\beta)} + \tau_{\alpha\beta}\right] + \Theta_{\alpha\beta} \tag{39}$$

where

$$\tau_{\alpha\beta} \equiv \sigma^2 \left[\phi_{,\alpha}\phi_{,\beta} - \frac{1}{2}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}g_{\alpha\beta} - \mathcal{U}^\mu\phi_\mu \left(\mathcal{U}_{(\alpha}\phi_{,\beta)} - \frac{1}{2}\mathcal{U}^\nu\phi_{,\nu}g_{\alpha\beta}\right)\right] - \frac{1}{4}Gl^{-2}\sigma^4 F\left(kG\sigma^2\right)g_{\alpha\beta} \tag{40}$$

and

$$\Theta_{\alpha\beta} \equiv K\left(g^{\mu\nu}\mathcal{U}_{[\mu,\alpha]}\mathcal{U}_{[\nu,\beta]} - \frac{1}{4}g^{\sigma\tau}g^{\mu\nu}\mathcal{U}_{[\sigma,\mu]}\mathcal{U}_{[\tau,\nu]}g_{\alpha\beta}\right) - \lambda\mathcal{U}_\alpha\mathcal{U}_\beta \tag{41}$$

8.2.2 Scalar field equations

Varying the action w.r.t. σ gives us the relationship between σ and $\phi_{,\alpha}$ We obtain:

$$-kG\sigma^2 F - \frac{1}{2}\left(kG\sigma^2\right)^2 F' = kl^2 h^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} \tag{42}$$

When varying with respect to our dynamical scalar field ϕ, we must take care - noting that ϕ makes an appearence in $\mathcal{S}_m$ through the effective metric $\tilde{g}^{\alpha\beta}$. We get for the scalar field ϕ equation of motion:

$$\left[\sigma^2 h^{\alpha\beta}\phi_{,\alpha}\right]_{;\beta} = \left[g^{\alpha\beta} + \left(1 + e^{-4\phi}\right)\mathcal{U}^\alpha\mathcal{U}^\beta\right]\tilde{T}_{\alpha\beta} \tag{43}$$

Using (42) we are able to eliminate σ and write (43) in terms of derivatives of ϕ only, with $\tilde{T}_{\alpha\beta}$ as a source term.

Given the form of (42), let us define a function $\mu(y)$ by

$$-\mu F(\mu) - \frac{1}{2}\mu^2 F'(\mu) = y \tag{44}$$

This gives for (43)

$$\left[\mu\left(kl^2 h^{\mu\nu}\phi_{,\mu}\phi_{,\nu}\right)h^{\alpha\beta}\phi_{,\alpha}\right]_{;\beta} = kG\left[g^{\alpha\beta} + \left(1 + e^{-4\phi}\right)\mathcal{U}^\alpha\mathcal{U}^\beta\right]\tilde{T}_{\alpha\beta} \tag{45}$$

We shall use this relation in order to find the behaviour in the MONDian limit.

8.3 Recovering the MONDian limit

We can simplify (45) when we have a symmetry from a collinearity of $\tilde{u}_\alpha$ (which appears in the definition of the energy-momentum tensor) with $\mathcal{U}_\alpha$. So $\tilde{u}_\alpha = e^\phi\mathcal{U}_\alpha$. Thus it follows from (34), that

$$\tilde{g}_{\alpha\beta} + \tilde{u}_\alpha\tilde{u}_\beta = e^{-2\phi}\left(g_{\alpha\beta} + \mathcal{U}_\alpha + \mathcal{U}_\beta\right) \tag{46}$$

Substituting this into (38) allows us to simplify (45) as

$$\left[\mu\left(kl^2 h^{\mu\nu}\phi_{,\mu}\phi_{,\nu}\right)h^{\alpha\beta}\phi_{,\alpha}\right]_{;\beta} = kG\left(\tilde{\rho} + 3\tilde{p}\right)e^{-2\phi} \tag{47}$$

In the static case (assuming the sources do not change through time), we set all the temporal derivatives in (47) to zero just as we did in the derivation of (14) for RAQUAL. This means $h^{\alpha\beta}\phi_{,\alpha} \rightarrow$

$g^{\alpha\beta}\phi_{,\alpha}$. The nonrelativistic approximation in addition allows us to replace $g^{\alpha\beta} \to \eta^{\alpha\beta}$ and $e^{-2\phi} \to 1$. We assume in addition that the pressure is negligble compared to the density $\tilde{p} \ll \tilde{\rho}$. Thus we obtain

$$\nabla \cdot \left[\mu \left(kl^2 \left(\nabla\phi\right)^2\right) \nabla\phi\right] = kG\tilde{\rho} \tag{48}$$

And considering the form of Poisson's equation in comparison to (48), we can make an identification between the newtonian potential and ϕ.

$$k^{-1}\mu|\nabla\phi| = \mathcal{O}\left(\nabla\Phi_N|\right) \tag{49}$$

And thus we have obtained (9), the AQUAL limit, only with a suitable reinterpretation of the function μ.

8.4 Successes

TeVeS is currently the most successful gravitational formulation of MOND. It can be shown [4] that with suitable choice of the function F, the appropriate general relativistic and cosmological limits can be reached. These coincide with what we expect. In addition to these successes, TeVeS explains the increase in light deflection by galactic lenses. TeVeS retains these successes without violating any of our principles for a coherent theory of gravity. TeVeS is *not* a preferred frame theory - TeVeS is fully covariant and has been well formulated in the language of relativity. TeVeS satisfies the equivalence principle - all non-gravitational forces (matter terms) enter in the matter part of the action and all move on the manifold whose metric is the effective metric $\tilde{g}_{\alpha\beta}$. TeVeS boasts no superluminal propagation of scalar field, vector field, or gravity waves. These success however, come at the cost of increased complexity and structure, as well as the addition of three arbitrary parameters and one function whose values and form respectively depend of the phenomenology we wish to examine.

9 Conclusions

9.1 Closing thoughts on the MOND paradigm

As a phenomenological scheme for explaining the flatness of the galactic rotation curve, MOND has been successful. However the principle behind it requires a change to our cherished law of inertia. This ansatz naturally places us in a precarious position. Indeed, MOND is worthless beyond its power to absolve the galactic rotation curve predicament. Thus, creative solutions were proposed in an attempt to reforumlate MOND, and to secure its phenomenological features in a specific limiting arena of a broader, more complete theory of gravity.

Due to the strict requirements of any modern theory of gravity, the mathematical instruments behind these *creative solutions* are indeed more stringent than they are inventive. The scalar and vector field extensions to Einstein gravity are designed precisely to implement MONDian dynamics, without violating any dearly-held *principle*, or upsetting our requirements. There is no physical meaning or interpretation to any of the scalar or vector fields. It was mentioned earlier that the MOND law itself (3) is simply an ad-hoc conjuration which serves only to flatten the rotation curve. In the same way, it should be stated that the underlying action-level characteristics of all the discussed theories of gravity bring functions and fields into play, whose base physical interpretations are not apparant. Of course MOND should not be considered to lie on the same logical footing with its successor theories. Certainly, the addition of such complexity has helped to recast the MONDian law into a complete theory of gravity, and procured many worth principles. But in terms of physical *motivation* of TeVeS' underlying components, TeVeS has developed MOND no further. The question of *why* is never addressed. Of course, we could pass the buck, and say that TeVeS itself belongs to a limit in a higher, more encompassing theory, which is able to well justify its ansatz. But by itself, while TeVeS is not an effective theory, it is a well-manipulated one.

9.2 Dark matter vs. MOND

The convincing argument that Dark Matter makes in astrophysics must be mentioned. From a theoretical standpoint, the dark matter hypothesis rests on a solid footing in astrophysics and cosmology. Recently even, experiments have managed to catch up to the theory with considerable success. Weak lensing on large scales has been successful in mapping out the filamental dark-matter structure of parts of the sky. Observations of the Bullet Cluster [14] (figures (3), (9.2)), show direct empirical proof of the presence of dark matter. It has been shown even more recently, that even TeVeS is unable to account enough to an increased deflection angle in weak lensing. Thus, even TeVeS requires dark matter.

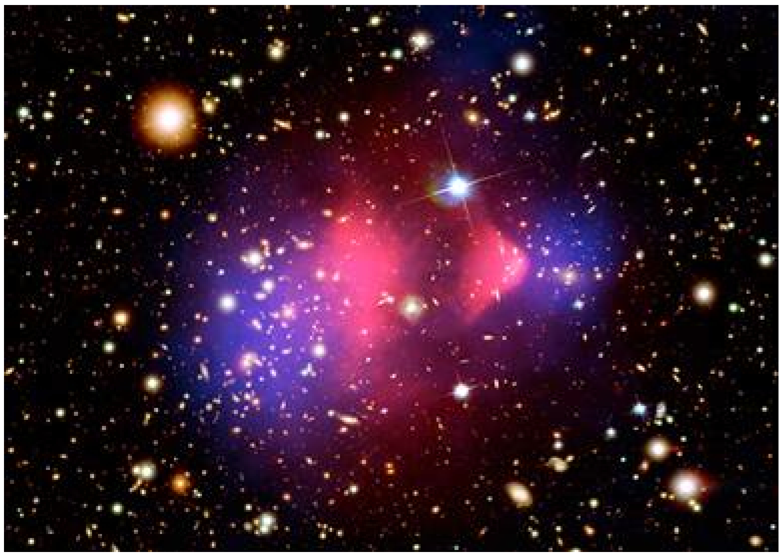

Figure 3: Nails in the MOND propositions's coffin: The Bullet Cluster is a collision of two galctic clusters. The red shaded region shows the baryonic (visible) matter component to the system, as observed directly from our telescopes. The blue shaded region however, shows the matter distribution as calculated by the weak lensing of light sources beyond the Bullet Cluster. This shows that there is a sizable quantity of matter in the bullet cluster *beyond* the visible. Were the MOND/TeVeS paradigm correct, the visible matter would be total matter contribution.[14]

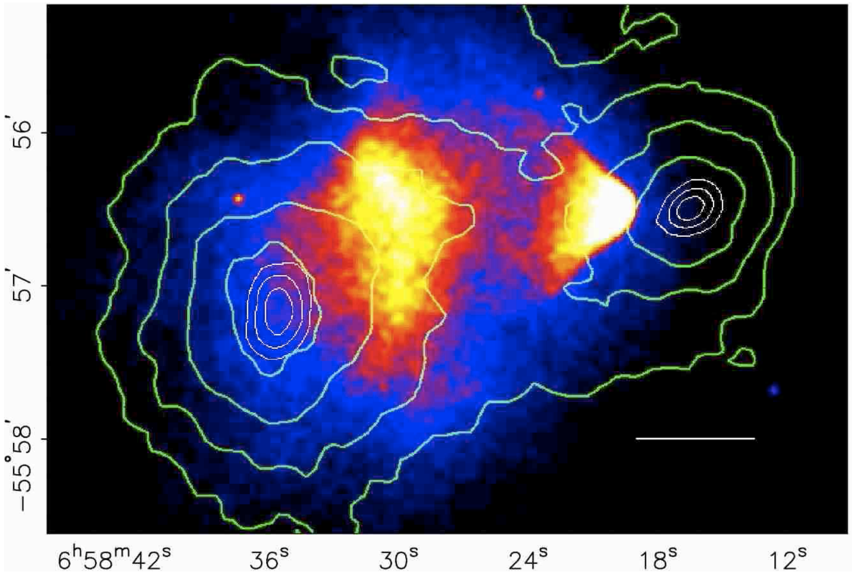

Figure 4: The bullet cluster with the black to white colours indicating the density distribution of the visible matter, and the contour lines indicating the matter density distribution as calculated from weak lensing.[14]

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